Newton-Raphson Fractals

The basic algorithm

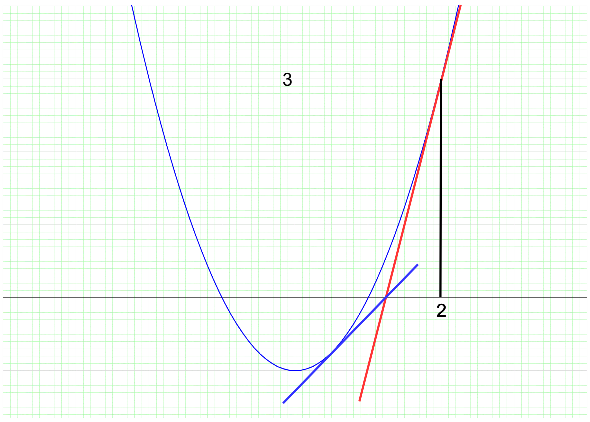
Suppose you want to find the square roots of 1. This amounts to finding a number *x* whose square is equal to 1. This means solving the equation

$$x^2 = 1 \tag{1}$$

or, alternatively $$f(x) = x^2 - 1 = 0 \tag{2}$$

Now the function $f(x) = x^2 - 1$ is a parabola which cuts the X axis in two places at +1 and -1. These are the roots that we seek.

In order to home in on these roots, we can proceed as follows. First we make a guess at the root. Let us guess $x = 2$. At this point, the graph is sloping steeply upwards and the value of the function at this point is 3 (2^2 - 1). A simple rule in calculus tells us that the gradient of this particular function is equal to $2x$ so the gradient at this point is actually +4. This is indicated by the red line.



If we trace this line back down to the X axis we can see that it will cut the axis at the point $x = 2 - 3/4 = 1.25$. This is obviously a better approximation to the root than the one than we first started with. If we repeat this process several times we can generate better and better approximations to the actual root.

In general, we shall suppose that if *x* is an approximation to a root of the equation then

$$x' = x - \frac{f(x)}{f'(x)} \tag{3}$$

is a better approximation (where $f'(x)$ is the gradient of the function $f(x)$ at the point *x*).

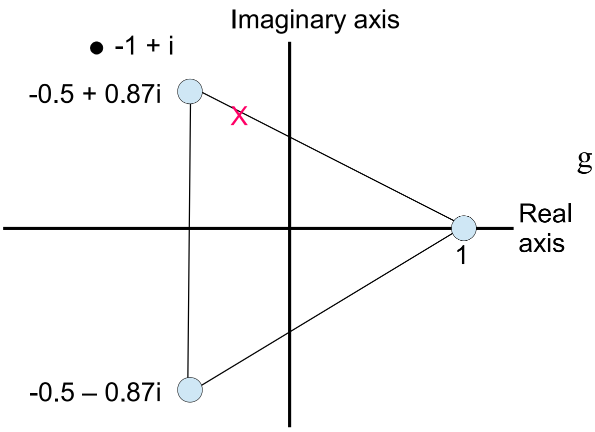
Suppose that we had started with a guess that $x = 0.5$. The value of $f(x)$ at $x = 0.5$ is -0.75 (0.5^2 - 1) and the gradient is +1 (2 × 0.5). Applying formula (3) we find that a better approximation to the root is $0.5 - \frac{-0.75}{1} = 1.25$. This is indicated by the blue line. I think it should be obvious that, if we start with a positive guess, whether it is smaller of larger than the actual root, eventually we will end up homing in on the positive root.

If, on the other hand, we start with the guess that $x = -2$, $f(x) = 3$ and $f'(x) = -4$. Applying formula (3) we get $-2 - \frac{3}{-4} = -1.25$ which is moving closer to the other root, namely -1. Likewise, starting with -0.5, we will also end up at the negative root.

Of course, if we start at $x = 0$, then the gradient is zero and the algorithm no longer works.

The complex version of the algorithm

Now the truly wonderful thing about complex numbers (numbers of the form $a + ib$ where i is the square root of minus one) is that anything which works with real numbers works with complex numbers too. Let us try to find the three cube roots of 1. These are 1, -0.5 + 0.87i and -0.5 – 0.87i and they form an equilateral triangle in the complex plane.



To see how the algorithm works using complex numbers, lets start with the guess $z = -1 + i$. (We tend to use z for complex variables), indicated by the black dot. Hopefully this will home in on the upper left root.

First we must calculate $f(z)$ and $f'(z)$ at this point. (N.B. Just as the gradient of the real function was $2x$, the gradient of the complex function will be $2z$.) So:

$$f(z) = (-1 + i)^2 - 1 = (1 - 2i + 1) - 1 = 1 - 2i$$
$$f'(z) = -2 + 2i$$

Now we have got to calculate the expression

$$(-1 + i) - \frac{1 - 2i}{-2 + 2i}$$

Doing division with complex numbers is a bit tricky but you can take it from me that this expression boils down to

$$(-1 + i) - \frac{-3 + i}{4} = -0.25 + 0.75i$$

This point is indicated by the red cross and is a good bit closer to the root than the black dot.

You will be glad to hear that from now on we will let the computer handle all the complex arithmetic. Every pixel on the screen will represent a starting point for our iteration z_0. The computer will calculate formula (3) over and over again until the modulus[1] of $f(z)$ is less than a small figure. When it has found a root, the program will see if it is a root which has already been found; if not, it adds it to a list and allocates a colour to that root. It then colours the original pixel which represents the original starting point z_0 with that colour.

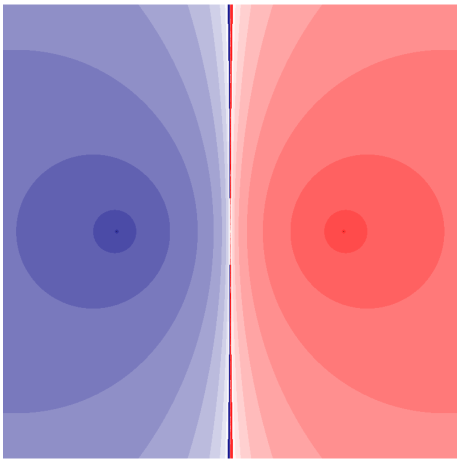
The big question now is what sort of patterns will the computer generate?

Lets start with the simplest function: $z^2 - 1$. Just like its real counterpart, this function has just 2 roots +1 and -1. It seems intuitively obvious that any value of z_0 which has a positive real part will home in on the positive root and any value of z_0 which has a negative real part will home in on -1. It is not so obvious what happens to values of z_0 which are purely imaginary. By symmetry, the point cannot home in on either of the two roots so there are only three options available: either it ends up at a stable fixed value; alternatively it could enter a fixed periodic

1 The modulus of a comple number $z = a + ib$ is equal to the distance of z from the origin and is equal to $\sqrt{(a^2+b^2)}$

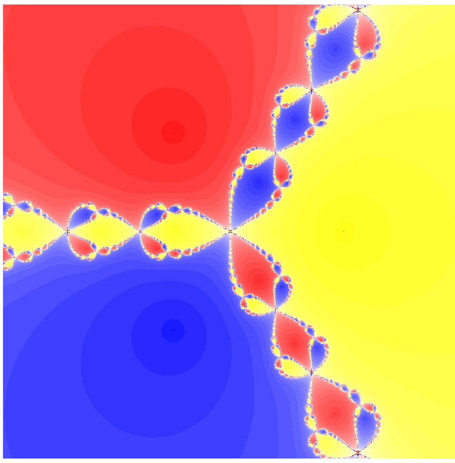
cycle; or it could spiral off to infinity. What we expect, therefore, is for the plane to have just two colours divided down the imaginary axis by a black line representing those values which do not home in on a root.

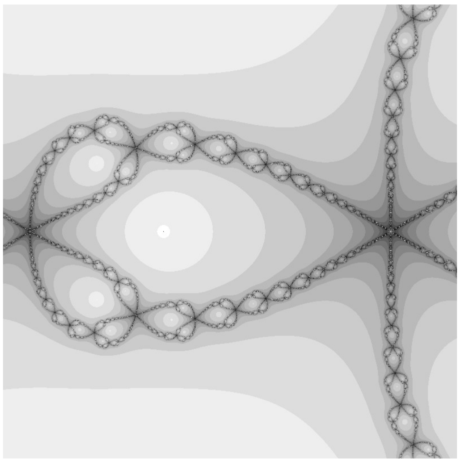
Here it is (the different shades are determined by how many iterations are needed to reach the root):



The next question is this: what do you think the map will look like for the function $z^3 - 1$?

If you have never seen this before, there is no way you could possibly guess. Here it is:





You may have guessed that the plane would be divided into three equal segments but there is no way you could foresee that the boundary between the segments was broken up into an endless chain of beads. One of the beads is shown above in greater detail.

It is an astonishing fact that every single point which on the boundary between two of the roots is also on the boundary of the third root as well.

The boundary is fractal because the more you magnify it, the more detail comes into view. Unlike the Mandelbrot set, however, the detail is always the same. It is 'turtles all the way down'.

Now for the really big question: why is the boundary fractal and not simply straight?

The first thing to say is that there is no a priori reason why the boundary should be fractal. You can easily invent an algorithm that simply moves towards the nearest root – what one might call the Buriden's ass algorithm[2]. So it must be something to do with the details of the Newton-Raphson algorithm.

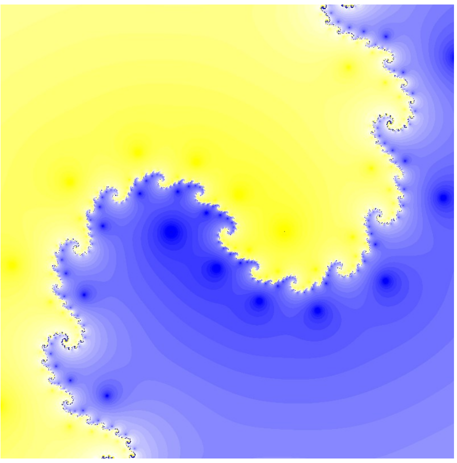
Obviously the origin must be symmetrical with respect to the three roots and is therefore a point at which all three basins meet. Now consider the three major branch points at the other end

2 Buriden's Ass refers to a hypothetical situation in which a donkey is placed equidistant from two or more heaps of hay.

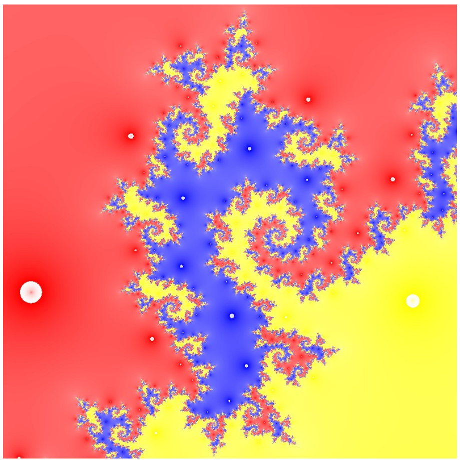
of the three beads which meet at the origin. After one iteration, these three points jump to the origin. Next consider a circle of points very close to this point. These will jump to a circle of points very close to the origin. It follows that this circle of points must include all three colours. It turns out that *every* point on the boundary eventually jumps to the origin after a finite number of iterations. It follows that every point on the boundary must be surrounded by all three colours and this is a sufficient condition for the boundary to be fractal. (With the Buriden's Ass algorithm, only the origin is surrounded by all three colours; all the other boundary points separate just two colours.)

The modified algorithm

On interesting way to modify the Newton-Raphson algorithm is to multiply the $f(z)/f'(z)$ term by a constant (which is not too far from unity) before subtracting it from z. The constant (which we shall call R) can itself be a complex number. If R is real and less than 1, it makes no difference to the fractal – it just slows down the calculations. Keeping the modulus of R to be 1 but using a complex number introduces a pleasing twist into the two fractals we have met so far.

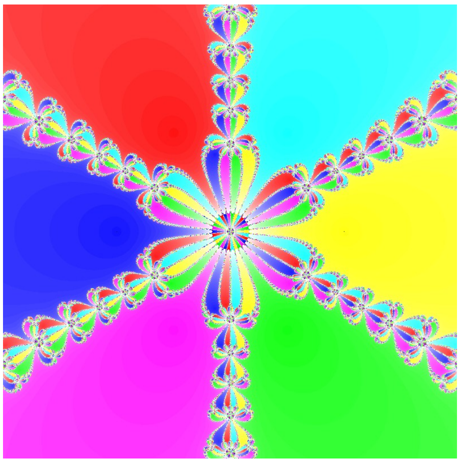


$z^2 = 1$ with 45^0 twist

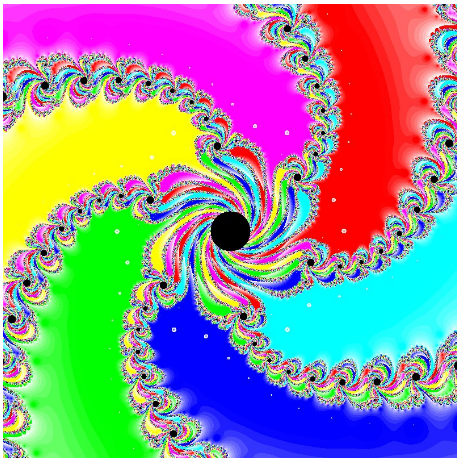


$z^3 = 1$ with 45^0 twist

Here are a couple of nice fractals based on the equation $z^6 = 1$. (The 'black hole' in the second example is due to the fact that the iterated point has jumped out beyond an arbitrary limit set by the program. These points are deemed to have 'escaped' – but in reality the only point to truly escape is the origin.)



$z^6 = 1$



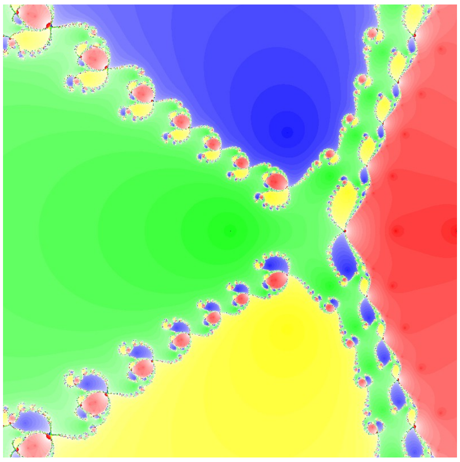
$z^6 = 1$ with 45^0 twist

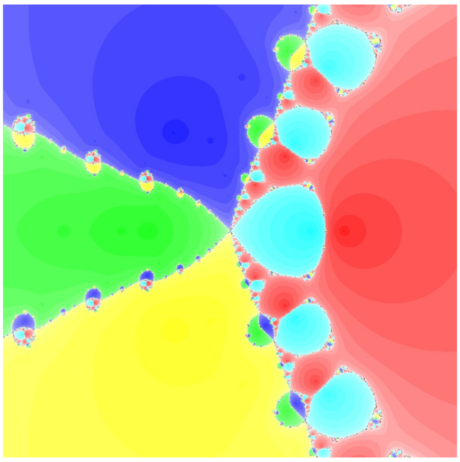
Other functions which include the roots of unity

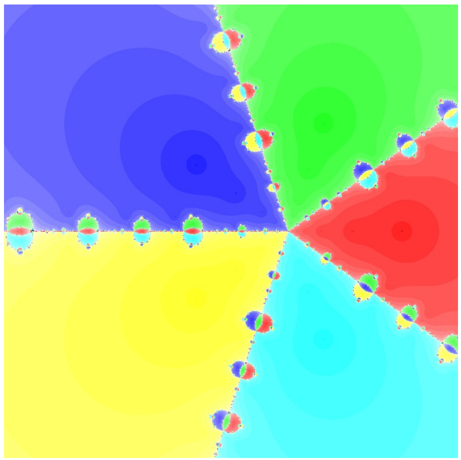
The equation z^n - 1 is not the only function which is equal to zero when $z^n = 1$. In fact any polynomial which has a factor $z^n - 1$ will do:

$$f(z) = (z^n - 1)\,g(z)$$

Of course, *g*(z*)* will have roots as well which will also appear in the fractal. Here are some examples in which *g*(*z*) is a polynomial:


$(z^3 - 1)(z^2 - 1)$


$(z^3 - 1)(1/z^2 - 2)$

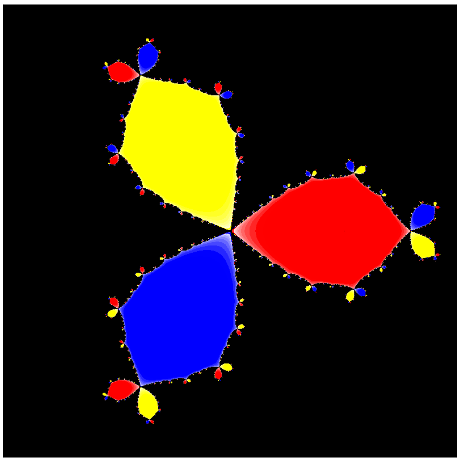

$(z^5 - 1)/z$

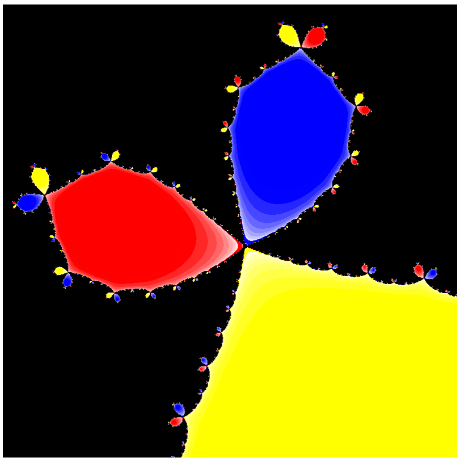
In the first case, although the equation is of order 5, there are only four roots because the root $z = 1$ is shared.

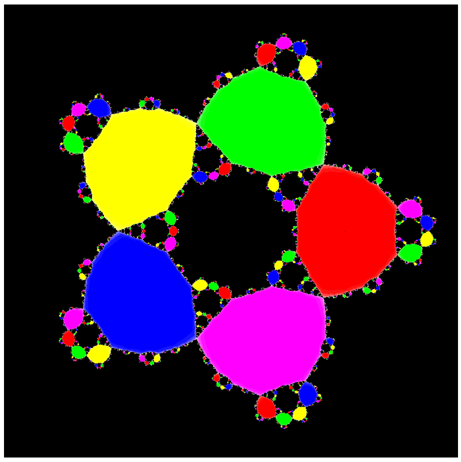
Here is another function. As before *g*(*z*) can be any function of *z*

$$f(z) = \frac{g(z)}{z^n - 1 + g(z)} - 1 \qquad (4)$$

but we shall restrict ourselves here to the simplest cases when *g*(*z*) is just a complex number R.

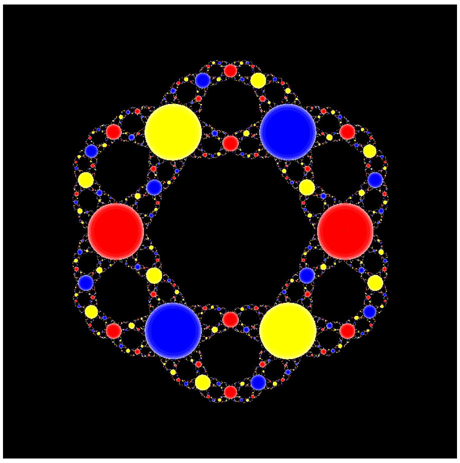

N = 3, R = 1


N = 3, R = 1 (detail)

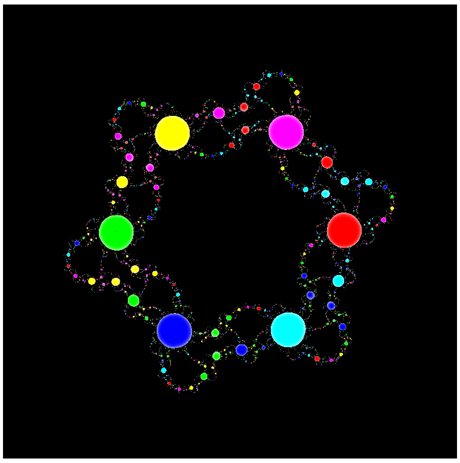

N = 5, R = 2

Owing to the reciprocal nature of the function, the great majority of starting values escape off to infinity. These are coloured black. Only those points fairly near to the a root converge.

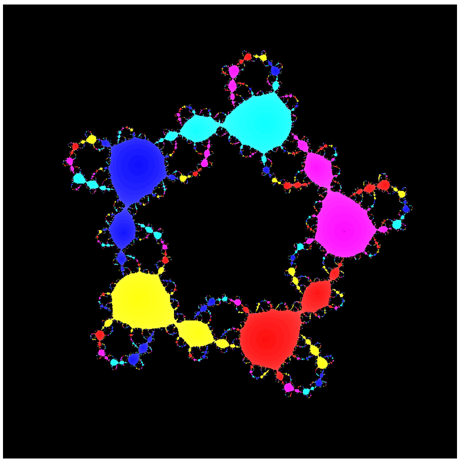
When *R* is negative or complex we get a rather pleasing set of fractals:



N = 3, R = -1

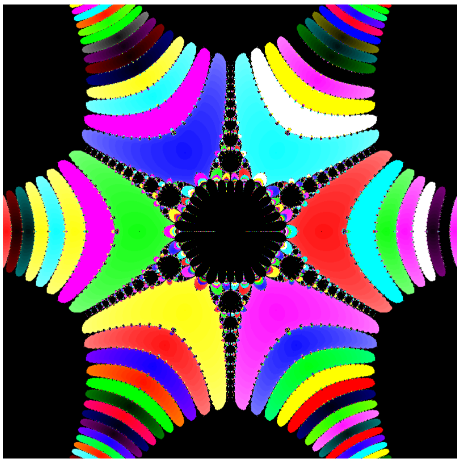


N = 6, R = i

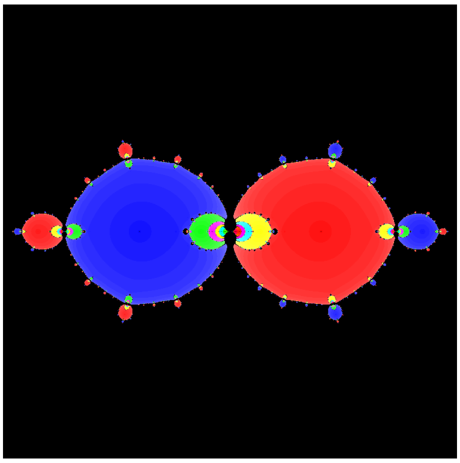


N = 5, R = 0.512 + i

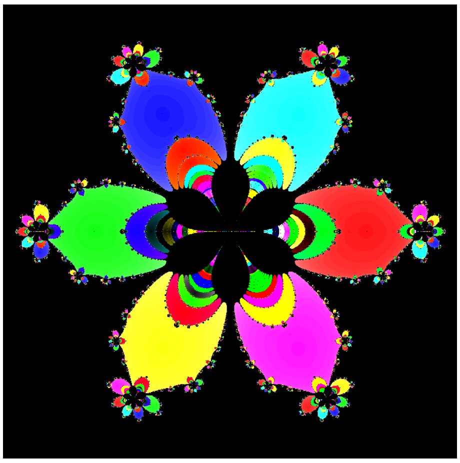
There are plenty of other functions which go to zero when $z = 1$. One of them is $\cos(\pi/2\ z^n)$. Whenever z^n 1 the expression boils down to $\cos(\pi/2)$ which is zero. This is what the NR fractal looks like for various values of n:



cos($\pi/2\ z^3$)



cos($\pi\ /2z$)

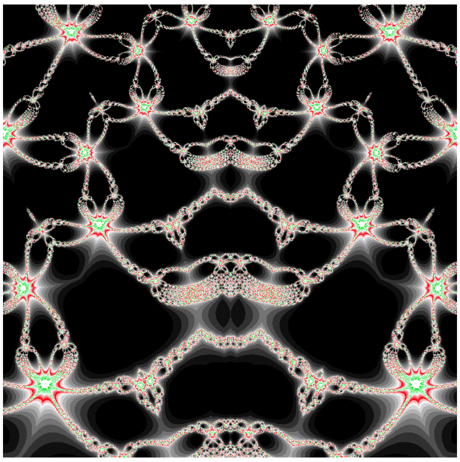


cos($\pi\ /\ 2z^3$)

When $n = 3$, the areas in red, blue and yellow converge on the three roots of unity. But because of the cyclic nature of the cosine function, other areas also converge on other solutions to the equation.

In the second and third examples, the exponent n is negative. The third example is a kind of inverse of the first.

Finally – because it is Christmas, here is a fractal generated by the equation $z^{3.7} = 1$



Merry Christmas!

15/12/2024